TWO COMMENTS ON DVORETZKY'S SPHERICITY THEOREM

BY

E. G. STRAUS

ABSTRACT

For any two positive integers k, l and any $\varepsilon > 0$ there exists an $N(k, l, \varepsilon)$ so that given any l convex bodies $C_1, \ldots, C_l$ symmetric about the origin in E^n with $n \geqq N$ there exists a subspace E^k so that each C_i intersects E^k, or has a projection into E^k, in a set which is nearly spherical (asphericity $< \varepsilon$). The measure of the totality of E^k which intersect a given body in E^n in a nearly ellipsoidal set is considered and an affine invariant measure is introduced for that purpose.

A convex set C in E^n which is centrally symmetric about the origin is said to have *asphericity*

$$\alpha(C) = 1 - \min_{x \in bdC} \|x\| / \max_{x \in bdC} \|x\|$$

where bd$C$ is relative to the subspace spanned by C. Dvoretzky [1] proved that: *For every positive integer k and every $\varepsilon > 0$ there exists a number $N(k,\varepsilon)$ (e.g., $N(k,\varepsilon) = \exp(2^{15}\varepsilon^{-2}k^2\log k)$), so that for $n \geqq N$, every convex body (compact convex set with non-empty interior) in E^n which is symmetric about the origin there exists a subspace E^k with $\alpha(C \cap E^k) < \varepsilon$.*

In a recent paper [2] Dvoretzky remarks that the same result holds if we consider the projection $C \mid E^k$ of C into E^k instead of $C \cap E^k$, since

$$\alpha(C \mid E^k) = \alpha(C^* \cap E^k)$$

where C^* is the polar body of C. However he states as an unsolved question whether there is an $N'(k,\varepsilon)$ so that for $n \geqq N'$ there exists an E^k for which both

$$\alpha(C \cap E^k) < \varepsilon \text{ and } \alpha(C \mid E^k) < \varepsilon.$$

To give an affirmative answer to this question we prove the following.

THEOREM. *For each pair of positive integers k, l and every $\varepsilon > 0$ there exists an $N(k,l,\varepsilon)$ so that for $n \geqq N$ and any l-tuple of convex bodies $C_1, \ldots, C_l$ in E^n symmetric about the origin, there exists a subspace E^k so that*

$$\alpha(C_i \cap E^k) < \varepsilon \qquad i = 1, \ldots, l.$$

Here $N(k,1,\varepsilon) = N(k,\varepsilon)$ and $N(k, l+1, \varepsilon) \leqq N(N(k,l,\varepsilon),\varepsilon)$.

Received February 6, 1964.

Proof. For $l = 1$ this is Dvoretzky's theorem. Assume the theorem true for $l\cdot$ Then for $n \geqq N(N(k,l,\varepsilon),\varepsilon)$ there exists an $E^{N(k,l,\varepsilon)}$ so that $\alpha(C_1 \cap E^{N(k,l,\varepsilon)}) < \varepsilon$ and by the induction hypothesis applied to $C_i' = C_i \cap E^{N(k,l,\varepsilon)}$, $i = 2,\dots,l+1$; there exists an $E^k \subset E^{N(k,l,\varepsilon)}$ so that $\alpha(C_i' \cap E^k) = \alpha(C_i \cap E^k) < \varepsilon$ for $i = 2,\dots, l$. On the other hand we have $\alpha(C_1 \cap E^k) \leqq \alpha(C_1 \cap E^{N(k,l,\varepsilon)}) < \varepsilon$ so the result holds with $N(k,l+1,\varepsilon) = N(N(k,l,\varepsilon),\varepsilon)$.

Dvoretzky's question is now answered in the affirmative for $C_1 = C$, $C_2 = C^*$, $l = 2$. The bound computed here grows very rapidly since it involves l-fold iteration of an already very rapidly increasing function of k and $1/\varepsilon$.

A second question raised in [2] can be answered in the negative. Dvoretzky proves that it is not possible to give a uniform positive lower bound for the Haar measure of the set of all k-planes ($k \geqq 2$) which intersect a convex body C in E^n in a set of asphericity $< \varepsilon$. His example is an ellipsoid of revolution with a very large axis on its axis of revolution. He asks therefore whether such a uniform lower bound could exist if asphericity is replaced by *unellipsoidality*, that is the minimum asphericity of all affine transforms of the set.

As an example of a body for which this is not the case we consider the union of two spherical caps:

$$x_1^2 + \dots + x_{n-1}^2 + (x_n - 1 + \delta)^2 \leqq 1,\ x_1^2 + \dots + x_{n-1}^2 + (x_n + 1 - \delta)^2 \leqq 1; \quad 0 < \delta < 1.$$

Every E^2 intersects C in a lens, which in terms of Cartesian coordinates (y_1, y_2) on E^2 can be given by

$$y_1^2 + (y_2 - r + \delta')^2 \leqq r^2,\ y_1^2 + (y_2 + r - \delta')^2 \leqq r^2.$$

Here

$$r^2 = 1 - (1-\delta)^2 + (r - \delta')^2$$

$$r - \delta' = (1-\delta)\cos\gamma$$

where γ is the angle between E^2 and the x_n-axis. Thus for δ sufficiently small we have δ'/r arbitrarily small for all γ outside an arbitrarily small neighborhood of $\pi/2$. Thus all we need is the following.

LEMMA. *The lens*

$$x^2 + (y - 1 + \delta)^2 \leqq 1,\ x^2 + (y + 1 - \delta)^2 \leqq 1;\ 0 < \delta < 1$$

has unellipsoidality $> 1/10 + O(\sqrt{\delta})$.

Proof. Because of the symmetry of the lens it suffices to consider diagonal transformations of the form $x' = x$, $y' = cy$. The radius in the x-direction remains $\sqrt{2\delta} + O(\delta)$ while the radius in the y-direction becomes $c\delta$. Thus, if the unellipsoidality is $\leqq 1/10 + O(\sqrt{\delta})$ we have $c\delta \leqq (10/9)\sqrt{2\delta} + O(\delta)$. Now the point $(\sqrt{\delta}, \delta/2 + O(\delta))$ on the lens goes into $(\sqrt{\delta}, c\delta/2 + O(\sqrt{\delta}))$ whose distance from the origin is

$$\sqrt{\delta + c^2\delta^2/4} + O(\delta) \leqq \sqrt{131\delta/81} + O(\delta) = \sqrt{131/162}\sqrt{2\delta} + O(\delta) \leqq (9/10)\sqrt{2\delta} + O(\delta)$$

which proves the lemma.

It may perhaps be argued that the question is not a well posed one since unellipsoidality is an affine invariant while the measure on the set of planes is not. Indeed it is easy to see that any neighborhood of an E^k in E^n can be transformed into a set of Haar measure $> 1 - \delta$ for any $\delta > 0$ by suitable stretching in the directions perpendicular to E^k. Hence for $n \geqq N(k,\varepsilon)$ any convex body C in E^k which is centrally symmetric about the origin is affine equivalent to a C' for which the Haar measure of all E^k so that $E^k \cap C'$ has unellipsoidality $< \varepsilon$ is greater than $1 - \delta$. By the same token, if C is not an ellipsoid, let β be the maximal unellipsoidality of $C \cap E^k$ for all E^k; then there is an affine equivalent C' of C so that the Haar measure of the E^k for which the unellipsoidality of $C' \cap E^k$ exceeds $\beta - \varepsilon$ is greater than $1 - \delta$.

Thus, in order to make the question more meaningful we should replace Haar measure by an affine invariant measure (such possibilities are indicated in [2]).

DEFINITION. Given a convex body C in E^n which is centrally symmetric about the origin. We define affine invariant measures $\mu_{n,k}(C;S_k)$ on sets S_k of k-subspaces as follows

(i) $\mu_{n,1}(C;S_1)$ is the Lebesgue measure of $\cup_{S_1}(C \cap E^1)$ divided by the Lebesgue measure of C.

(ii) $\mu_{n,k}(C;S_k) = \int \dots \int \chi(E_1^1,\dots,E_k^1)d\mu_{n,1}(C;E_1^1)\dots d\mu_{n,1}(C;E_k^1)$

where χ is 1 if $E_1^1,\dots,E_k^1$ lie in one of the E^k in S_k and 0 otherwise, and the integral is extended over all k-tuples $(E_1^1,\dots,E_k^1)$. It is now clear that for any affine transformation T we have $\mu_{n,k}(TC;TS_k) = \mu_{n,k}(C;S_k)$.

PROBLEM. *Does there exist a number $N(k,\varepsilon,\delta)$ so that for every convex body C symmetric about the origin in E^n with $n \geqq N$ the set S_k of E^k in E^n with unellipsoidality of $C \cap E^k$ less than ε satisfies $\mu_{n,k}(C,S_k) > 1 - \delta$?*

REFERENCES

1. Dvoretzky, A. 1961, Some results on convex bodies and Banach spaces, *Proc. Intl. Symposium on Linear Spaces*, Pergamon Press and Jerusalem Academic Press, pp. 123–160.

2. ———, 1963, Some near-sphericity results, *Proc. Symposia in Pure Math.* VII (Convexity), pp. 203–210.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES